Scaled fractional Fourier transform and its optical implementation

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The scaled fractional Fourier transform is suggested and is implemented optically by one lens for different values of ϕ and output scale. In addition, physically it relates the FRT with the general lens transform—the optical diffraction between two asymmetrically positioned planes before and after a lens. © 1997 Optical Society of America
Key words: Optics, Fourier transform.

The concept of fractional Fourier transform (FRT)[1,2] has been accepted and its applications have been found in optics.[3–10] Since its definition was first proposed, some variations have been created for its optical implementation,[11–19] but the integrals in these previous optical implementations always take the equivalent form of

$$F^{p}u(x)=\int_{-\infty}^{+\infty}u(x)\exp\left[\frac{i\pi}{\lambda f_1}\left(\frac{x^2+y^2}{\tan\phi}-\frac{2xy}{\sin\phi}\right)\right]\mathrm{d}x, \quad (1)$$

where $\phi = p\pi/2$, and $f_1 = f\sin\phi$ in Ref. 11, or $f_1 = f(1-\cos\phi)/\sin\phi$ in Ref. 16, or $f_1 = f_\rho \sin\phi$ in Ref. 18, and so on. In this article we implement the integral

$$F^{\phi}u(x)=\int_{-\infty}^{+\infty}u(x)\exp\left\{i\pi\left[\frac{x^2+(My)^2}{\tan\phi}-\frac{2x(My)}{\sin\phi}\right]\right\}\mathrm{d}x, \quad (2)$$

where $u(x)$ is the function to be transformed and M is the desired scale factor. In appearance, the above two integrals are similar; but in fact they are different in three respects: (1) the coefficients of the variables x and y in the exponent of Eq. (1) are always the same, but in Eq. (2) the coefficients are usually not the same; (2) Eq. (1) has the parameter f_1 that depends on the angle ϕ, but Eq. (2) contains the free parameter M; and (3) the optical implementation of the two integrals is not completely the same.

In previous implementations, the geometric configurations were symmetric (that is, the distance $z_1 = z_2$) unless a spherical wave was used for illumination, and regardless of whether the plane wave or the spherical wave was used, the integrals performed in these configurations were identical, i.e., Eq. (1). But in the implementation of Eq. (2), plane wave and asymmetric optical configurations are used; this can been seen in the following: The integral of Eq. (2), which we call scaled FRT, is not only closer to the Namias definition[1] than the integral of Eq. (1) from the mathematical point of view but it also may be more useful and convenient for optical information processing because the free parameter M introduced into the FRT integral makes the output scale of the transformation adjustable. Also it can been seen that when $M = 1$ and $\phi \to \pi/2$, the integral $F^{\phi}u(x)$ is consistent with the normal type of mathematical Fourier transform. Through investigation, we found that this integral must be implemented by the general lens transform shown in Fig. 1, where u and v denote input and output, respectively, and z_1 and z_2 express the distance. We expanded the input (say, a transparency) q times first, then located it at the input plane and illuminated it with a plane wave. The light wave on the output plane is determined by two steps of Fresnel diffraction:

$$v(y)=\int_{-\infty}^{+\infty}\exp\left[\frac{i\pi}{\lambda}\frac{(x_L-y)^2}{z_2}\right]\left\{\int_{-\infty}^{+\infty}u(x/q)\times\exp\left[\frac{i\pi}{\lambda}\frac{(x-x_L)^2}{z_1}\right]\mathrm{d}x\right\}\exp\left(-\frac{i\pi}{\lambda}\frac{{x_L}^2}{f}\right)\mathrm{d}x_L$$

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Received 2 January 1997; revised manuscript received 7 May 1997.
0003-6935/97/328490-03$10.00/0

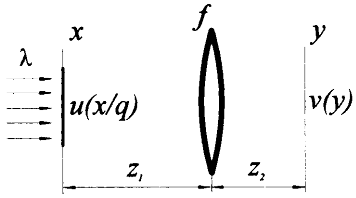


Fig. 1. Geometry of general lens transform.

$$= \int_{-\infty}^{+\infty} u(x/q) \exp\left[\frac{i\pi}{\lambda}\left(\frac{x^2}{z_1} + \frac{y^2}{z_2}\right)\right] \times \exp\left[-\frac{i\pi}{\lambda}\,\frac{(x/z_1 + y/z_2)^2}{1/z_1 + 1/z_2 - 1/f}\right] I \mathrm{d}x,$$

where λ and f are the wavelength and the focal length, respectively; x_L is the coordinate at the lens; and

$$I = \int_{-\infty}^{+\infty} \exp\left\{\frac{i\pi}{\lambda}\left[x_L\left(\frac{1}{z_1} + \frac{1}{z_2} - \frac{1}{f}\right)^{1/2} - \frac{x/z_1 + y/z_2}{(1/z_1 + 1/z_2 - 1/f)^{1/2}}\right]^2\right\} \mathrm{d}x_L.$$

Because the result of the integral I is a constant with nothing to with the variables x or y, after omitting all constant factors, we have

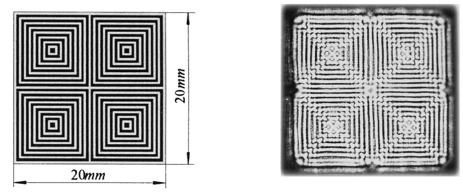
$$v(y) = \int_{-\infty}^{+\infty} u(x/q) \exp\left[\frac{i\pi}{\lambda}\left(\frac{x^2}{z_1} + \frac{y^2}{z_2}\right)\right] \times \exp\left[-\frac{i\pi}{\lambda}\,\frac{(x/z_1 + y/z_2)^2}{1/z_1 + 1/z_2 - 1/f}\right] \mathrm{d}x$$
$$= \int_{-\infty}^{+\infty} u(x/q) \times \exp\left[\frac{i\pi}{\lambda}\,\frac{(f - z_2)x^2 + (f - z_1)y^2 - 2fxy}{z_1 f + z_2 f - z_1 z_2}\right] \mathrm{d}x. \tag{3}$$

By letting

$$x/q = t,$$
$$\cos\phi = \frac{(f - z_1)^{1/2}(f - z_2)^{1/2}}{f}, \tag{4}$$

$$M = \left[\frac{(f - z_1)^{1/2}}{\lambda(f - z_2)^{1/2}(z_1 f + z_2 f - z_1 z_2)^{1/2}}\right]^{1/2}, \tag{5}$$

$$q = \left[\frac{\lambda(f - z_1)^{1/2}(z_1 f + z_2 f - z_1 z_2)^{1/2}}{(f - z_2)^{1/2}}\right]^{1/2} = \frac{1}{M}\,\frac{(f - z_1)^{1/2}}{(f - z_2)^{1/2}}, \tag{6}$$



a b

Fig. 2. Function and its scaled FRT: a, function to be transformed; b, optical experimental result of its scaled FRT. In the experiment, $\phi = 38°$, $M = 0.79$, $q = 1.2$, $f = 3900$, $z_1 = 978$, and $z_2 = 667$.

we have

$$v(y) = \int_{-\infty}^{+\infty} u(t) \exp\left\{i\pi\left[\frac{t^2 + (My)^2}{\tan\phi} - \frac{2t(My)}{\sin\phi}\right]\right\} \mathrm{d}t, \tag{7}$$

which is the integral of Eq. (2) that we want to implement (in deduction, all constant coefficients are dropped). As a consequence, we can use the general optical diffraction through a lens to implement the scaled FRT provided that the input is expanded q times according to Eq. (6) beforehand and the distances are determined by Eqs. (4) and (5):

$$z_1 = f - \lambda M^2 f^2 \sin\phi \cos\phi, \tag{8}$$

$$z_2 = f - \frac{1}{\lambda M^2 \tan\phi}. \tag{9}$$

As an example, we transform a two-dimensional function, whose pattern is illustrated in Fig. 2a, according to Eq. (2). In the practical operation, so that any dimensional confusion is avoided, all the length units are taken as millimeters (and are omitted in text for simplicity). In the experiment, we chose $\lambda = 0.0006328$, $\phi = 38°$, $f = 3900$, $M = 0.79$, and with Eqs. (8), (9), and (6) we valued the parameters $z_1 = 978$, $z_2 = 667$, and $q = 1.2$. Magnifying the object 1.2 times (with a scanner–computer–laser printer system) and allocating it at the x plane in Fig. 1, we obtained the experimental result on the y plane shown in Fig. 2b.

In the previous optical implementation, the coefficients of their variables are always the same regardless of whether illuminated with a plane wave or a spherical wave; and if f_1 is given [see Eq. (1)], a particular lens is suitable only for a particular value of ϕ unless two or more lenses are used.[20,21] But now we can use one lens for performing the FRT with different values of ϕ; meanwhile a desired output scale is available. In addition, as far as the physical aspect is concerned, we have related the FRT with the general lens transform—the optical diffraction between two asymmetrically positioned planes before and after a lens.

This research is supported by the National Foundation of Natural Science of the People's Republic China under grant 69477001.

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